MATHEMATICS

ON THE EXISTENCE OF A SUBINVARIANT MEASURE [1)]

BY

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Haar measure is invariant under the homeomorphisms induced by the group operation in the measure space. Consider instead the problem of finding a subinvariant measure for a locally compact space with respect to a set $\mathscr{G}$ of homeomorphisms. That is, we look for a measure λ such that $\lambda(GB) \leqq \lambda(B)$ for all $G \in \mathscr{G}$ and Borel sets B. Clearly the existence of such a measure when $\mathscr{G}$ is a group implies that λ is already invariant, so it is natural to consider semigroups $\mathscr{S}$ of homeomorphisms instead. Furthermore, for a monotone set function λ the relation $GB \supset B$ implies $\lambda(GB) \geqq \lambda(B)$, and it is therefore natural to require $GB \not\supset B$ for all $G \in \mathscr{S}$.

In this paper we take the underlying space to be the open unit interval I. The construction of the set function λ given below follows the construction of Haar measure for compact sets as described in [1, Ch. XI].

The problem of a subinvariant measure on an interval has arisen from an economic problem in the axiomatics of utility [2, 3]. The latter problem concerns choice between consumption programs each consisting of an infinite sequence of future consumption vectors. The points of I on which $\mathscr{S}$ operates are utility levels of these programs. The elements G of $\mathscr{S}$ represent the effect on utility levels of postponement of programs by a stated number of time units. Each G is labeled by that number and by the "momentary" utility levels associated with the consumption vectors inserted in the gaps created by postponement. The existence of a measure on I subinvariant for $\mathscr{S}$ signifies a certain lack of patience with regard to the time of availability of desirable goods.

Theorem 1. *Let $\mathscr{S}$ be a semi-group of homeomorphisms from I, the open unit interval, to I, having the properties*

(a) *that $GU \supset U$ never holds for an interval U of I and a $G \in \mathscr{S}$, and*

(b) *that for any given open interval U of I an arbitrary point of I can be covered by GU for some $G \in \mathscr{S}$.*

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Then there exists a real function λ defined on closed intervals D of I, finitely additive on intervals with disjoint interiors, positive on non-degenerate intervals, and such that $\lambda(D) \geqq \lambda(GD)$ for all $G \in \mathscr{S}$ and all $D \subset I$.

Proof: Fix a point p in I and let U be an open interval containing p. If D is a closed subinterval of I let

$$(D : U) = \min \{n \mid D \subset \bigcup_{i=1}^{n} U_i,\ U_i = G_i U,\ G_i \in \mathscr{S}\}.$$

Obviously, $(D : U) \geqq 1$, and it follows from the compactness of D that $(D : U)$ is finite. Define, for A fixed, closed and non-degenerate in I,

$$\lambda_U(D) = (D : U)/(A : U).$$

If the intervals $U_i = G_i U$, $G_i \in \mathscr{S}$, $i = 1, \ldots, n$, form a cover of A by n images of U and the intervals $A_j = G_j' \mathring{A}$, $G_j' \in \mathscr{S}$, $j = 1, \ldots, n'$, a cover of D by n' images of $\mathring{A}$, then clearly $U_{ji} = G_j' G_i U$ is a cover of D by $n'n$ images of U, with $G_j' G_i \in \mathscr{S}$ for all j, i. Hence $(D : U) \leqq (D : \mathring{A}) \cdot (A : U)$, and, if D is non-degenerate,

$$(1) \qquad O < \frac{1}{(A : \mathring{D})} \leqq \frac{(D : U)}{(A : U)} = \lambda_U(D) \leqq (D : \mathring{A})$$

Let Φ be the set of functions f defined for closed intervals D in I and such that $O \leqq f(D) \leqq (D : \mathring{A})$. Provide Φ with the topology of convergence on finite sets $\{D_1, D_2, \ldots, D_i\}$ [4, p. 92]. Then Φ is compact, by Tychonoff's theorem on the compactness of a product of compact spaces [4, p. 143].

Let $\Lambda(U) = \{\lambda_V \mid U \supset V, p \in V\}$. It is straightforward to verify [1, p. 255] that the family of all sets $\Lambda(U)$ has the property that any finite subfamily $\{\Lambda(U_1), \ldots, \Lambda(U_n)\}$ has a non-empty intersection. Since Φ is compact there is therefore a function λ in $\bigcap_{p \in U} \overline{\Lambda(U)}$. For D non-degenerate, (1) implies $\lambda(D) > 0$.

To show that λ is finitely additive we use two lemmas.

Lemma 1. *If $\{U_n\}$, $n = 1, 2, \ldots$, is a nested sequence of neighborhoods of p converging to p, then $\lim_{n \to \infty} (A : U_n) = \infty$.*

Proof of Lemma 1. Clearly $(A : U_n)$ is a non-decreasing function of n. Suppose $(A : U_n) \leqq N$ for all n for some fixed integer N. Choose N disjoint open intervals A_i, $i = 1, \ldots, N$, in A. By premise (b) of Theorem 1 we can find open intervals U_i' about p such that $G_i A_i = U_i'$ for some $G_i \in \mathscr{S}$. Let $U_0 = \bigcap_{i=1}^{N} U_i'$. Then $p \in U_0$ and, by premise (a) of Theorem 1, no A_0 with $GU_0 = A_0$ for some G can contain an A_i. Hence N images of U_0 under $\mathscr{S}$ could not cover A, which contradicts the premise.

Lemma 2. *If U is a neighborhood of p and D and E are closed intervals with disjoint interiors and a common end point, then*

$$-1/(A : U) + \lambda_U(D) + \lambda_U(E) \leqq \lambda_U(D \cup E) \leqq \lambda_U(D) + \lambda_U(E).$$

Proof of Lemma 2. The second inequality holds because the union of minimal coverings of D and E is a covering of $D \cup E$, perhaps not minimal. The first inequality holds because the latter covering of $D \cup E$ can be turned into a minimal covering of $D \cup E$ by removing at most one interval that covers the common endpoint of D and E.

To prove additivity of λ, notice that $\lambda \in \overline{\Lambda(U)}$, for all neighborhoods U of p, implies that there is for any finite set of closed intervals $\{D_1, \ldots, D_N\}$ a nested sequence U_n converging to p, such that $\lambda_n = \lambda_{U_n} \in \Lambda(U_n)$ and $\lambda_n(D_k)$ converges to $\lambda(D_k)$ for $k = 1, \ldots, N$. For let U_n' be a sequence of neighborhoods of p converging to p. Since $\lambda \in \bigcap_{p \in U} \overline{\Lambda(U)}$, we have $\lambda \in \bigcap_{n=1}^{\infty} \overline{\Lambda(U_n')}$. Then, for any given finite set $\{D_1, \ldots, D_N\}$, there is for all n a $U_n'' \subset U_n'$ such that $| \lambda_{U_n''}(D_k) - \lambda(D_k) | < 1/n$, $k = 1, \ldots, N$. Since the sequence U_n'' converges to p, it contains a nested subsequence U_n converging to p such that λ_n converges to λ on the set $\{D_1, \ldots, D_N\}$.

For the set $\{D_1, \ldots, D_N\}$ we now take $\{D, E, D \cup E\}$ of Lemma 2. Then, given $\varepsilon > 0$, there is an n_ε such that $n > n_\varepsilon$ implies

$$| \lambda_n(D \cup E) - \lambda(D \cup E) | < \varepsilon/3,$$
$$| \lambda_n(D) - \lambda(D) | < \varepsilon/3, \qquad | \lambda_n(E) - \lambda(E) | < \varepsilon/3.$$

From these inequalities it follows that

$$-\varepsilon + \lambda_n(D) + \lambda_n(E) - \lambda_n(D \cup E) \leqslant \lambda(D) + \lambda(E) - \lambda(D \cup E) \leqslant$$
$$\leqslant \lambda_n(D) + \lambda_n(E) - \lambda_n(D \cup E) + \varepsilon.$$

But by Lemma 2,

$$1/(A : U_n) \geqslant \lambda_n(D) + \lambda_n(E) - \lambda_n(D \cup E) \geqslant 0.$$

Therefore, for all n,

$$-\varepsilon \leqslant \lambda(D) + \lambda(E) - \lambda(D \cup E) \leqslant 1/(A : U_n) + \varepsilon.$$

By Lemma 1, $1/(A : U_n)$ tends to zero. Since ε is arbitrary, λ is additive.

To check that $\lambda(GD) \leqslant \lambda(D)$ it is enough to check the same condition for arbitrary λ_U. Now $(GD : U) \leqslant (D : U)$ because a minimal covering of D by sets $G_i U$ gives rise to a covering, not necessarily minimal, of GD by sets $GG_i U$. The desired result follows on division by $(A : U)$.

Corollary. *λ is zero on one-point sets.*

Proof: If D is a one-point set, $(D : U) = 1$ for all U. The corollary follows by Lemma 1.

Theorem 2. *Any interval function λ of Theorem 1 is continuous in the sense that, if D_n is a nested sequence of closed non-degenerate intervals, converging to a fixed point q, then $\lim_{n \to \infty} \lambda(D_n) = 0$.*

Proof [1]). For any $\varepsilon > 0$, there are in I non-degenerate intervals having λ-measure at most ε. To see this take an interval having finite positive measure M and partition it into at least $M\varepsilon^{-1}$ non-degenerate intervals. One of these must have measure at most ε. Let E_ϵ be a non-degenerate interval of measure at most ε. Then, for some $G \in \mathscr{S}$, GE_ϵ contains q in its interior by premise (b) of Theorem 1, and $\lambda(GE_\epsilon) \leqslant \lambda(E_\epsilon) \leqslant \varepsilon$. For n sufficiently large $D_n \subset GE_\epsilon$ so $\lambda(D_n) \leqslant \varepsilon$.

[1]) We are indebted to R. Strichartz for this simple proof.

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